Complex Numbers Overview

A (Mostly) Programming-Free Introduction to Complex Numbers for the Cave of Programming Advanced C++ Course

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What Are Complex Numbers?

Complex numbers are widely used in mathematics, physics and, on occasion, computer programming. We'll explore how complex numbers work in this document, from a starting point of basic algebra. It'll also help if you have some knowledge of trigonometry functions.

In basic arithmetic, it's not possible to find a square root of -1; that is, a number which gives -1 when multiplied by itself. But in basic mathematics we happily make use of the square root of -1. We define a number, usually represented by the letter i as being equal to the square root of -1.

$$i = \sqrt{-1}$$

We say that i is an *imaginary* number. We can multiply i by other numbers to form other imaginary numbers, for example $4i$, $32.5i$ and so on.

Visualising Complex Numbers

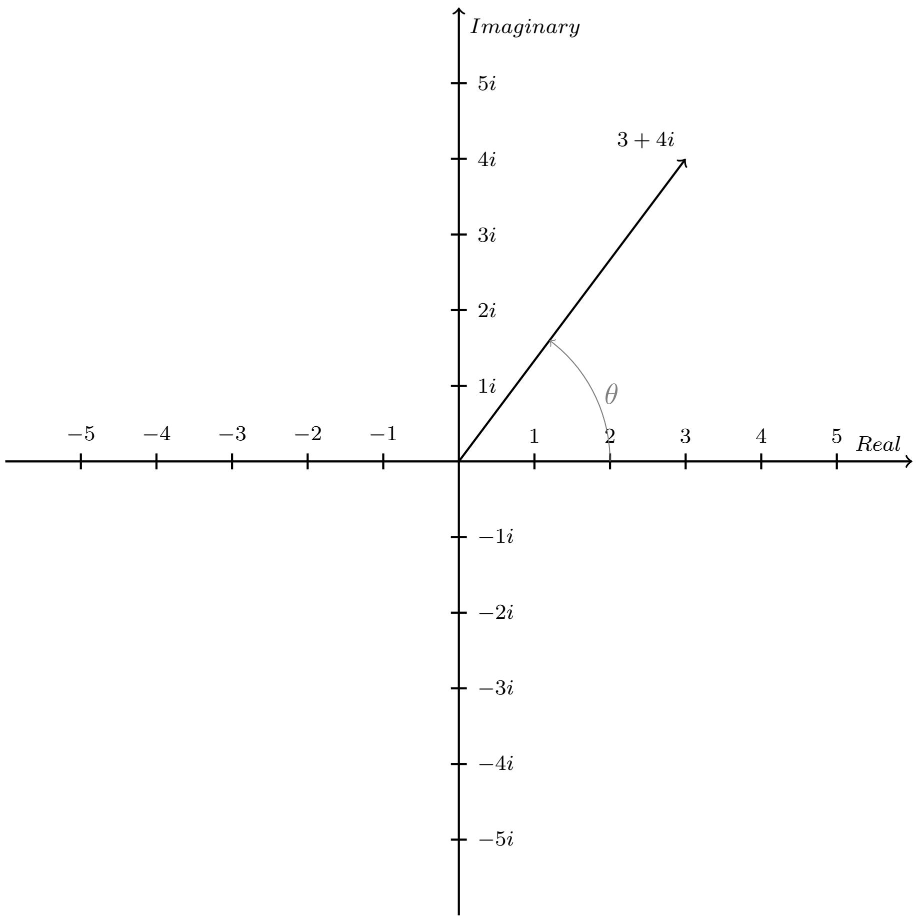
In contrast to imaginary numbers, the non-imaginary numbers that we're familiar with are described as *real* numbers. We can form composite numbers, called *complex numbers*, by adding together real and imaginary numbers. For example:

$$3 + 2i$$

$$1.3 - 6.2i$$

.... and so on. We think of these entities as single numbers, which have a real part and an imaginary part.

We can visually represent complex numbers using a Cartesian coordinate system. Usually the vertical axis is imaginary, while the horizontal axis is real. We call this the *complex plane*. Let's represent $3 + 4i$ on the complex plane.



The complex number is a point on the complex plane; here I've drawn an arrow from the origin to the point, to make it easier to see. I've also marked the angle between the real axis and this arrow with the Greek letter θ (theta).

The length of this arrow – in other words, the distance from the origin to the point that represents the complex number on the plane – is known as the *magnitude* of the complex number. The angle θ is known as the complex number's *argument*.

Magnitude and Argument

Let's define a complex number $z = 3 + 4i$. Referring to the above diagram and applying Pythagoras' theorem, we see that the magnitude of of α, written as $|z|$ is:

$$|z| = |3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

The magnitude is always real.

We can make use of the trigonometric function *arctan*, also known as inverse tan or tan^{-1}, to find the angle θ. This function takes the ratio between the opposite and adjacent sides of a triangle and converts it into the angle between the adjacent side and the hypotenuse, which is exactly what we want.

$$\theta = \tan^{-1}\frac{Im(z)}{Re(z)} = \tan^{-1}\frac{4}{3} = tan^{-1}1.3333 = 53.13°$$

... where $Im(z)$ is the imaginary part of z and $Re(z)$ is the real part.

There's a problem here, which is that arctan can't tell from a simple ratio between the lengths of two sides, such as 1.3333 in the above equations, which quadrant the complex number actually lies in. Sometimes we need that information, so we have to adopt a definition of arctan that performs slightly different calculations depending on whether each of the two lengths ($Im(z)$ and $Re(z)$ in this case) are positive or negative.

Fortunately, if you're writing a computer program, many languages provide you with a version of arctan called atan2 which takes two arguments; namely, the two lengths in question – in this case, $Im(z)$ and $Re(z)$ – and correctly returns an angle in the right quadrant.

If we're given the magnitude r and argument θ of a complex number and want to write down the number in the form $x + iy$, we can use the *cos* and *sin* trig functions to make the necessary calculations, also available in most computer programming languages. *cos* gives us the projection of the line in the above diagram onto the horizontal axis, while *sin* gives us its projection onto the vertical axis.

$$x + iy = r(cos\theta + isin\theta)$$

Multiplication

We can multiply two complex numbers using ordinary algebra. Remember that a real number cannot become imaginary unless it's multiplied by an imaginary number, and an imaginary number can only become real if it's multiplied by an imaginary number. When two i's multiply, they turn into -1, a real number.

$$(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) = \alpha_1\alpha_2 + i\alpha_1\beta_2 + i\alpha_2\beta_1 + i^2\beta_1\beta_2 = \alpha_1\alpha_2 - \beta_1\beta_2 + i(\alpha_1\beta_2 + \alpha_2\beta_1)$$

The strange thing is, when you multiply two complex numbers like this, the resulting number has a magnitude equal to the magnitudes of the two original numbers multiplied, and its argument is equal to the *sum* of the arguments of the original two.

Conjugation

The *conjugate* of a complex number is the same number, except that we change the sign of the imaginary part. The conjugate of z is written $\overline{z}$. The notation z^* is also used, especially in physics.

$$\overline{z} = z^* = \overline{x + iy} = x - iy$$

This means that the magnitude of a complex number is equal to itself multiplied by its complex conjugate.

$$\overline{z}z = (x - iy)(x + iy) = x^2 - i^2y^2 + iyx - iyx = x^2 + y^2 = |z|$$

Why??? Why All These Crazy Numbers??

A lot of the utility of complex numbers comes from the fact that, referring to the diagram we saw earlier, we can think of imaginary numbers as lying at 90°to real numbers. Moreover, as we've seen, multiplying complex a complex number by another complex number effectively rotates the first by the argument of the second. For example, multiplying 1 by i gives us i, which lies at 90° to 1 in the complex plane. Multiplying i again by i gives us -1; another 90° rotation, and so on.

These facts give complex numbers a variety of applications. For instance, we can use complex numbers to represent a wave of circularly polarised light, which we can think of as consisting of two out-of-phase light waves at 90° to each other.