Section 7.2 Addition and Subtraction Formulas

Addition and Subtraction Formulas

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

EXAMPLE: Find the exact value of each expression.

(a) $\cos 75^\circ$ (b) $\cos \frac{\pi}{12}$

Solution:

(a) Notice that $75^\circ = 45^\circ + 30^\circ$. Since we know the exact values of sine and cosine at45° and 30°, we use the addition formula for cosine to get

$$\begin{aligned}\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}\sqrt{3}-\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

(b) Since $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, the subtraction formula for cosine gives

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}\sqrt{3}+\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

EXAMPLE: Find the exact value of $\sin 15^\circ$.

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Solution: Notice that $15^\circ = 45^\circ - 30^\circ$. Since we know the exact values of sine and cosine at45° and 30°, we use the addition formula for sine to get

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}\sqrt{3}-\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

EXAMPLE: Find the exact value of the expression $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$.

Solution: We recognize the expression as the right-hand side of the addition formula for sine with $s = 20^\circ$ and $t = 40^\circ$. So we have

$$\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ = \sin(20^\circ + 40^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

EXAMPLE: Find the exact value of the expression $\cos 10^\circ \sin 20^\circ + \sin 10^\circ \cos 20^\circ$.

Solution: We recognize the expression as the right-hand side of the addition formula for sine with $s = 10^\circ$ and $t = 20^\circ$. So we have

$$\cos 10^\circ \sin 20^\circ + \sin 10^\circ \cos 20^\circ = \sin 10^\circ \cos 20^\circ + \cos 10^\circ \sin 20^\circ = \sin(10^\circ + 20^\circ) = \sin 30^\circ = \frac{1}{2}$$

EXAMPLE: Find the exact value of the expression $\cos 10^\circ \cos 20^\circ - \sin 10^\circ \sin 20^\circ$.

Solution: We recognize the expression as the right-hand side of the addition formula for cosine with $s = 10^\circ$ and $t = 20^\circ$. So we have

$$\cos 10^\circ \cos 20^\circ - \sin 10^\circ \sin 20^\circ = \cos(10^\circ + 20^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

EXAMPLE: Find the exact value of the expression $\sin 87^\circ \sin 27^\circ + \cos 87^\circ \cos 27^\circ$.

Solution: We recognize the expression as the right-hand side of the subtraction formula for cosine with $s = 87^\circ$ and $t = 27^\circ$. So we have

$$\sin 87^\circ \sin 27^\circ + \cos 87^\circ \cos 27^\circ = \cos 87^\circ \cos 27^\circ + \sin 87^\circ \sin 27^\circ = \cos(87^\circ - 27^\circ) = \cos 60^\circ = \frac{1}{2}$$

EXAMPLE: Prove the following identities

(a) $\cos\left(\frac{\pi}{2} - x\right) = \sin x \quad$ and $\quad \cos\left(\frac{\pi}{2} + x\right) = -\sin x$

(b) $\sin\left(\frac{\pi}{2} \pm x\right) = \cos x$

EXAMPLE: Prove the following identities

(a) $\cos\left(\frac{\pi}{2}-x\right)=\sin x$ and $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$

(b) $\cos\left(x-\frac{\pi}{2}\right)=\sin x$

(c) $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and $\sin\left(\frac{\pi}{2}+x\right)=\cos x$

(d) $\sin\left(x-\frac{\pi}{2}\right)=-\cos x$

Solution:

(a) We have

$$\cos\left(\frac{\pi}{2}-x\right)=\cos\frac{\pi}{2}\cos x+\sin\frac{\pi}{2}\sin x=0\cdot\cos x+1\cdot\sin x=\sin x$$

and

$$\cos\left(\frac{\pi}{2}+x\right)=\cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x=0\cdot\cos x-1\cdot\sin x=-\sin x$$

In short,

$$\cos\left(\frac{\pi}{2}\pm x\right)=\cos\frac{\pi}{2}\cos x\mp\sin\frac{\pi}{2}\sin x=0\cdot\cos x\mp 1\cdot\sin x=\mp\sin x$$

(b) Since cosine is an even function, we have

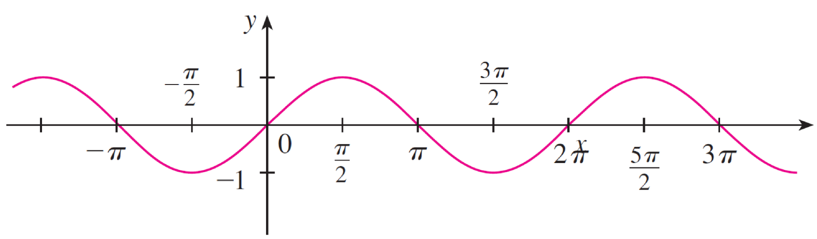
$$\cos\left(x-\frac{\pi}{2}\right)=\cos\left(-\left(\frac{\pi}{2}-x\right)\right)=\cos\left(\frac{\pi}{2}-x\right)=\sin x$$

(c) We have

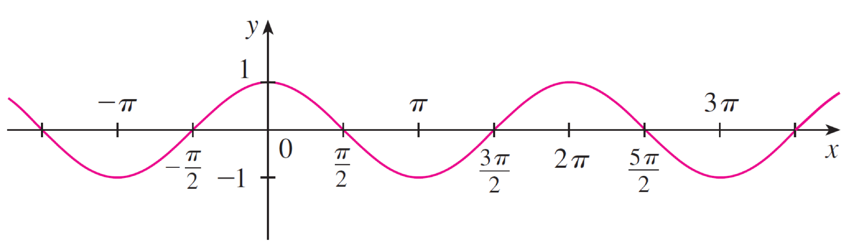
$$\sin\left(\frac{\pi}{2}\pm x\right)=\sin\frac{\pi}{2}\cos x\pm\cos\frac{\pi}{2}\sin x=1\cdot\cos x\pm 0\cdot\sin x=\cos x$$

(d) Since sine is an odd function, we have

$$\sin\left(x-\frac{\pi}{2}\right)=\sin\left(-\left(\frac{\pi}{2}-x\right)\right)=-\sin\left(\frac{\pi}{2}-x\right)=-\cos x$$



$y=\sin x$



$y=\cos x$

Sums of Sines and Cosines

If A and B are real numbers, then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos \phi = \frac{A}{\sqrt{A^2 + B^2}} \qquad \text{and} \qquad \sin \phi = \frac{B}{\sqrt{A^2 + B^2}}$$

EXAMPLE: Express $3 \sin x + 4 \cos x$ in the form $k \sin(x + \phi)$.

Solution: By the preceding theorem,

$$k = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

The angle ϕ has the property that $\sin \phi = \frac{4}{5}$ and $\cos \phi = \frac{3}{5}$. Using a calculator, we find $\phi \approx 53.1°$. Thus

$$3 \sin x + 4 \cos x \approx 5 \sin(x + 53.1°)$$